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The interpolation solution of the second basic plane problem of the dynamics of elastic solids☆

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ABSTRACT

The second basic plane problem of the dynamics of elastic bodies is considered in the Muskhelishvili formulation, when the known boundary displacements are replaced by interpolation time polynomials and the known initial conditions are replaced by polyharmonic functions, which interpolate the initial conditions in a region with a finite number of interpolation nodes. In this case a solution of the problem, called here the interpolation solution, is possible. It must satisfy the dynamic equations and interpolate the boundary displacements and initial displacements and velocities. This solution is constructed in the form of a polynomial and is reduced to solving a series of boundary-value problems for determining the coefficients of this polynomial.

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1. Formulation of the problem

Suppose D is a finite region of the XOY plane, which is a cross section of an elastic cylinder Ω with generatrices perpendicular to XOY. We will denote the boundary of D by C. The cylinder is subjected to a plane time-varying deformation, the same for all sections parallel to the XOY plane, parallel to the same plane. If $\boldsymbol{a} = (u, v)$ is the displacement vector of points of the region D, where

$$u = u(x, y, t), \quad \upsilon = \upsilon(x, y, t), \quad (x, y) \in D, \quad t \in [t_0, T]$$

then, provided that there are no mass forces, the dynamic equations for the components of the displacement vector have the form[1]

$$(\lambda + \mu)\partial_x\theta + \mu\Delta u = \rho\partial_{tt}u, \quad (\lambda + \mu)\partial_y\theta + \mu\Delta\upsilon = \rho\partial_{tt}\upsilon; \quad \theta = \partial_x u + \partial_y \upsilon. \tag{1.1}$$

The constants λ and μ are the Lamé coefficients. We formulate the problem of obtaining the displacement vector, the coordinates of which satisfy Eqs (1.1), when the following boundary values are known

$$(u(x, y, t_j), \upsilon(x, y, t_j))\big|_{(x, y) \in C} = (\tilde{u}_j(s), \tilde{\upsilon}_j(s)), \quad j = 0, 1, \ldots, n \tag{1.2}$$

where s is the arc abscissa of the curve C, $s \in [0, l]$, at the nodal points $t_j \in [t_0, T] (j = 0,1,\ldots n)$, the values of the initial velocity at a finite number of points of the region: $\boldsymbol{V}(x_k, y_k, t_0) = (V_k^1, V_k^2)(k = 1, \ldots, m)$ and the values of the initial displacement at a finite number of internal points of the region: $\boldsymbol{a}(x_{k'}, y_{k'}, t_0) = (a_{k'}^1, a_{k'}^2)(k' = 1, \ldots, m')$.

2. Interpolation of the boundary displacements

We will initially seek the displacements $\breve{u}(x, y, t)$, $\breve{\upsilon}(x, y, t)$ which satisfy the dynamic equations and boundary displacements, in the form

$$\breve{u}(x, y, t) = \sum_{k=0}^{n} u_k(x, y)t^k, \quad \breve{\upsilon}(x, y, t) = \sum_{k=0}^{n} \upsilon_k(x, y)t^k \tag{2.1}$$

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We note immediately that, from boundary conditions (1.2), we can further obtain the values

$$u_k(x,y)\big|_{(x,y)\in C} = \hat{u}_k(s),\quad \upsilon_k(x,y)\big|_{(x,y)\in C} = \hat{\upsilon}_k(s),\quad k = 0,1,\ldots,n,$$

since the systems

$$\sum_{k=0}^{n}\hat{u}_k(s)t_j^k = \tilde{u}_j(s),\quad j = 0,1,\ldots,n$$

and

$$\sum_{k=0}^{n}\hat{\upsilon}_k(s)t_j^k = \tilde{\upsilon}_j(s),\quad j = 0,1,\ldots,n$$

with Vandermond determinant as the principal determinant of each system have a unique solution.

We now substitute the components of the displacement vector in the form (2.1) into Eqs (1.1) and equate coefficients for all powers of t. We obtain a system of equations for the coefficients $u_k(x,y)$ and $\upsilon_k(x,y)$

$$\partial_x[(\lambda+2\mu)(\partial_x u_k+\partial_y\upsilon_k)]-\partial_y[\mu(\partial_x\upsilon_k-\partial_y u_k)] = \rho(k+1)(k+2)u_{k+2}$$

$$\partial_y[(\lambda+2\mu)(\partial_x u_k+\partial_y\upsilon_k)]+\partial_x[\mu(\partial_x\upsilon_k-\partial_y u_k)] = \rho(k+1)(k+2)\upsilon_{k+2}$$

where

$$u_{n+1}(x,y)\equiv u_{n+2}(x,y)\equiv\upsilon_{n+1}(x,y)\equiv\upsilon_{n+2}(x,y)\equiv 0,\quad (x,y)\in D$$

If we multiply both sides of the second equation of the system by $i/2$ and add to the corresponding parts of the first equation, multiplied by ½ the previous system can be written in the form

$$\partial_{\bar{z}}[(\lambda+2\mu)(\partial_x u_k+\partial_y\upsilon_k)+i\mu(\partial_x\upsilon_k-\partial_y u_k)] = \frac{\rho(k+1)(k+1)}{2}[u_{k+2}+i\upsilon_{k+2}] \tag{2.2}$$

Here we have introduced the complex variable $z=x+iy$, and derivatives with respect to the complex variables will, as usual, be taken to mean

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y),\quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

Note that we have changed from independent variables x and y to the independent variables $z=x+iy$ and $\tilde{z}=x-iy$, related to x and y by a linear non-degenerate transformation. (It is easy to verify that $z=x+iy$ and $\tilde{z}=x-iy$ are independent of one another by calculating $\partial z/\partial\bar{z}$ and $\partial\bar{z}/\partial z$.) All further calculations will be carried out in the new variables.

We will solve Eqs (2.2) successively, beginning with $k=n$, gradually reducing the number k. For $k=n$, $n-1$ we have

$$(\lambda+2\mu)(\partial_x u_k+\partial_y\upsilon_k)+i\mu(\partial_x\upsilon_k-\partial_y u_k) = f_k(z)$$

where $f_k(z)$ is a function analytic in the region D. Separating the real and imaginary parts and taking into account the expression for the derivative with respect to the complex variable z, we obtain

$$\partial_z(u_k+i\upsilon_k) = \frac{\mathrm{Re}f_k(z)}{2(\lambda+2\mu)}+i\frac{\mathrm{Im}f_k(z)}{2\mu} = \frac{f_k+\overline{f_k}}{4(\lambda+2\mu)}+\frac{f_k-\overline{f_k}}{4\mu}$$

Consequently,

$$u_k+i\upsilon_k = \frac{\lambda+3\mu}{4\mu(\lambda+2\mu)}\int f_k(z)dz-\frac{\lambda+\mu}{4\mu(\lambda+2\mu)}z\overline{f_k(z)}+\overline{g_k(z)}$$

where $g_k(z)$ is a function analytic in D. If we now introduce the notation

$$\frac{\lambda+\mu}{2(\lambda+2\mu)}\int f_k(z)dz\equiv\phi_k(z),\quad -2\mu g_k(z)\equiv\psi_k(z),\quad \frac{\lambda+3\mu}{\lambda+\mu}\equiv\kappa$$

we obtain the following representation of the complex plane displacement, like when solving the second basic problem of the plane theory of elasticity[1]

$$-2\mu(u_k+i\upsilon_k) = -\kappa\phi_k(z)+z\overline{\phi_k'(z)}+\overline{\psi_k(z)}$$

The boundary condition

$$[-\kappa\phi_k(z)+z\overline{\phi_k'(z)}+\overline{\psi_k(z)}]\big|_{z\in C} = -2\mu(\hat{u}_k(s)+i\hat{\upsilon}_k(s)),\quad s\in[0,l]$$

will be the same as when solving the second basic problem of the plane theory of elasticity.[1] If the curve C is specified using a function of the length of the arc, having continuous derivatives up to the third order, the functions $\phi_k(z)$ and $\psi_k(z)$, analytic in D, which are continuously extendable together with $\phi'_k(z)$ to the boundary C, can be reestablished in the region D using the well-known approach described in Ref. 1.

For $k \le n-2$ the method of solving Eq. (2.2) is practically the same except that an additional known term appears on the right-hand sides

$$(\lambda+2\mu)(\partial_x u_k+\partial_y \upsilon_k)+i\mu(\partial_x \upsilon_k-\partial_y u_k) = $$
$$= f_k(z)+\frac{\rho(k+1)(k+2)}{2}\int[u_{k+2}+i\upsilon_{k+2}]d\bar{z}$$

where $f_k(z)$ is a function analytic in the region D. Separating the real and imaginary parts and integrating over z, we obtain

$$-2\mu(u_k+i\upsilon_k) = -\kappa\phi_k(z)+z\overline{\phi'_k(z)}+\overline{\psi_k(z)}+\Phi_k(z,\bar{z})$$

where

$$\Phi_k(z,\bar{z}) = -2\mu(k+1)(k+2)S[u_{k+2}+i\upsilon_{k+2}]$$
$$S[u+i\upsilon] = k_1\int\mathrm{Re}[\int(u+i\upsilon)d\bar{z}]dz+ik_2\int\mathrm{Im}[\int(u+i\upsilon)d\bar{z}]dz$$
$$k_1 = \frac{\rho}{4(\lambda+2\mu)},\quad k_2 = \frac{\rho}{4\mu} \tag{2.3}$$

Note that $\Phi_k(z,\tilde{z})$ is a known function since the quantities $u_{k+2}+i\upsilon_{k+2}$ have already been obtained. To determine the functions $\phi_k(z)$ and $\psi_k(z)$, that are analytic in D, we have a boundary condition as when solving the second basic problem of the plane theory of elasticity

$$[-\kappa\phi_k(z)+z\overline{\phi'_k(z)}+\overline{\psi_k(z)}]\Big|_{z\in C} =$$
$$= -2\mu(\hat{u}_k(s)+i\hat{\upsilon}_k(s))-\Phi_k(z(s),\overline{z(s)}),\quad s\in[0,l]$$

After substituting all the coefficients $u_k(x,y)$, $v_k(x,y)(k=0,1,\ldots,n)$ we obtain, from formula (2.1), the displacements $\breve{u}(x,y,t)$, $\breve{v}(x,y,t)$, which satisfy Eqs (1.1) and the boundary conditions, but which do not satisfy the initial displacements and velocities.

We will show that, by construction, all the functions $u_k(x,y)$ and $v_k(x,y)$ are polyharmonic functions, and the order of harmonicity increases by unity on changing from $u_k(x,y)$ and $v_k(x,y)$ to $u_{k-2}(x,y)$ and $v_{k-2}(x,y)$.

We first note that the functions $u_n(x,y)$, $v_n(x,y)$, $u_{n-1}(x,y)$ and $v_{n-1}(x,y)$ are biharmonic in D, since they can be represented in the form of the real or imaginary parts of functions of the form $f_1(z)+\bar{z}f_2(z)$, where $f_j(z)$, $j=1, 2$ are analytic functions.

In fact,

$$2\mu u_j(x,y) = \mathrm{Re}Z_j^{(1)}(z),\quad 2\mu\upsilon_j(x,y) = \mathrm{Im}Z_j^{(2)}(z)$$
$$Z_j^{(1)}(z) = \kappa\phi_j(z)-\bar{z}\phi'_j(z)-\psi_j(z),$$
$$Z_j^{(2)}(z) = \kappa\phi_j(z)+\bar{z}\phi'_j(z)+\psi_j(z);\quad j = n, n-1$$

Note also that the analytic coefficients for $\bar{z}$ in the representations of the functions u_j and v_j ($j=n$, $n-1$) differ only in sign.

We will consider the action of the additive operator (2.3) on the m-harmonic function. Suppose

$$u(x,y) = \mathrm{Re}\left[\sum_{k=0}^{m-1}\bar{z}^k f_k(z)\right],\quad \upsilon(x,y) = \mathrm{Im}\left[\sum_{k=0}^{m-1}\bar{z}^k g_k(z)\right] \tag{2.4}$$

It can be shown that

$$\mathrm{Re}S[u+i\upsilon] = \mathrm{Re}[\Sigma_-+\Sigma_f+\Sigma_+],\quad \mathrm{Im}S[u+i\upsilon] = \mathrm{Im}[\Sigma_-+\Sigma_g-\Sigma_+]$$
$$\Sigma_- = \sum_{k=0}^{m-1}\bar{z}^k\frac{k_1-k_2}{4}\int\int[f_k(z)-g_k(z)]dzdz$$
$$\Sigma_h = \sum_{k=0}^{m-1}\bar{z}^{k+1}\frac{k_1+k_2}{2(k+1)}\int h_k(z)dz,\quad h = f, g$$
$$\Sigma_+ = \sum_{k=0}^{m-1}\bar{z}^{k+2}\frac{k_1-k_2}{4(k+1)(k+2)}[f_k(z)+g_k(z)]$$

It is obvious that, when $f_{m-1}(z)=-g_{m-1}(z)$, the coefficients of $\bar{z}^{m+1}$ of representations $ReS[u+iv]$ and $ImS[u+iv]$ vanish; moreover, the coefficients of $\bar{z}^m$ of the representations of the same functions will differ only in sign.

Hence, the interpolation solution $\breve{u}(x,y,t)$, $\breve{v}(x,y,t)$ obtained is the $([n/2]+2)$-harmonic function.

Since the solution having the form of a polynomial in powers of t, which takes specified values for values of the variable $t_j (j=0,1,\ldots,n)$ for fixed coordinates x and y of a point lying on the boundary of the region, is an interpolation polynomial, to estimate the components of the error of the interpolation of the boundary values of the displacement vector $\mathbf{r}(t)=(r_1(t), r_2(t))$ one can use the corresponding estimate of the residual term.[2] Thus, if the solution of the problem in the classical formulation for a specified boundary point (x,y) is the boundary displacement vector $\boldsymbol{a}(t)=(u(t), v(t))$, where $u(t)$ and $v(t)$ are $(n+1)$-times differentiable functions, we have

$$|r_1(t)| \le \frac{1}{(n+1)!}\max_{t_0\le t\le T}\left|u^{(n+1)}(t)\right|\prod_{k=0}^{n}|t-t_k|$$

$$|r_2(t)| \le \frac{1}{(n+1)!}\max_{t_0\le t\le T}\left|v^{(n+1)}(t)\right|\prod_{k=0}^{n}|t-t_k| \tag{2.5}$$

When constructing the interpolation solutions, the boundary displacements

$$\breve{u}(x(s), y(s), t_j),\quad \breve{v}(x(s), y(s), t_j),\quad s\in[0,l]$$

at the $(n+1)$-th instant of time, can be calculated accurately and, consequently, if these functions are differentiable, the values of the functions

$$\frac{d}{ds}\breve{u}(x(s), y(s), t),\quad \frac{d}{ds}\breve{v}(x(s), y(s), t) \tag{2.6}$$

when $t=t_j$ $(j=0,1,\ldots,n)$ can also be calculated accurately for each value of the arc parameter $s\in[0,l]$. Since, for each value of $s=s_0$, the functions (2.6) are polynomials in t with known values at the $(n+1)$-th node t_j, inequalities of the type (2.5) can also be used to estimate the errors in interpolating the derivatives of the solution along the length of the arc.

As an example we will consider the simplest problem when the region D is the unit circle and the displacements on its boundary are specified at three instants of time.

Suppose

$$t_0=0,\quad t_1=1,\quad t_2=2$$

$$\tilde{u}_0(\theta)\equiv 0,\quad \tilde{v}_0(\theta)\equiv\delta,\quad \tilde{u}_1(\theta)=(r-1)\cos\theta$$

$$\tilde{v}_1(\theta)=(r-1)\sin\theta,\quad \tilde{u}_2(\theta)\equiv\epsilon,\quad \tilde{v}_2(\theta)\equiv 0$$

where $\theta\in[0,2\pi]$ is the polar angle. This means that, at the initial instant of time, the arc is shifted in the direction of the OY axis, at the instant corresponding to $t=1$, it is uniformly compressed, and at the instant corresponding to $t=2$, it is shifted in the direction of the OX axis.

The interpolation solution of this problem, which satisfies the dynamic equation and the specified boundary displacements, has the form

$$\breve{u}(x,y,t)+i\breve{v}(x,y,t) = i\delta-(1-|z|^2)\frac{\rho(\epsilon+i\delta)}{2(\lambda+3\mu)}+z(1-|z|^2)\frac{\rho(r-1)}{4(\lambda+2\mu)}-$$

$$-\frac{\epsilon+3i\delta-4(r-1)z}{2}t+\frac{\epsilon+i\delta-2(r-1)z}{2}t^2,\quad x^2+y^2\le 1,\quad t\in[0,2]$$

3. Interpolation of the initial conditions

In order to interpolate the initial displacements and velocity, we will seek a complex solution of the initial problem in the form

$$w(x,y,t) = \breve{w}(x,y,t)+\tilde{w}(x,y,t)$$

$$w(x,y,t) = u(x,y,t)+iv(x,y,t),\quad \breve{w}(x,y,t) = \breve{u}(x,y,t)+i\breve{v}(x,y,t)$$

$$\tilde{w}(x,y,t) = \tilde{u}(x,y,t)+i\tilde{v}(x,y,t)$$

where the displacements $\breve{u}(x,y,t)$, $\breve{v}(x,y,t)$ are the same as those obtained in the previous section, i.e., they satisfy the dynamic equations and the interpolation boundary conditions, while the displacements $\tilde{u}(x,y,t)$, $\tilde{v}(x,y,t)$ satisfy the dynamic equations (1.1), are equal to zero on the boundary of the plate at specified instants of time t_j ($j=0, 1, \ldots, n$), and

$$\begin{aligned}
\tilde{u}(x_{k'}, y_{k'}, t_0) &= a^1_{k'} - \breve{u}(x_{k'}, y_{k'}, t_0)\\
\tilde{\upsilon}(x_{k'}, y_{k'}, t_0) &= a^2_{k'} - \breve{\upsilon}(x_{k'}, y_{k'}, t_0), \quad k' = 1, \ldots, m'\\
\partial_t\tilde{u}(x_k, y_k, t_0) &= V^1_k - \partial_t\breve{u}(x_k, y_k, t_0)\\
\partial_t\tilde{u}(x_k, y_k, t_0) &= V^2_k - \partial_t\breve{\upsilon}(x_k, y_k, t_0), \quad k = 1, \ldots, m
\end{aligned} \tag{3.1}$$

We will seek the displacements $(\tilde{u}, \tilde{\upsilon})$ in the form

$$\begin{aligned}
\tilde{u}(x, y, t) &= \sum_{k=0}^{n} \tilde{u}_k(x, y)t^k + \sum_{j=1}^{l} P^1_j(x, y)t^{n+j}\\
\tilde{\upsilon}(x, y, t) &= \sum_{k=0}^{n} \tilde{\upsilon}(x, y)t^k + \sum_{j=1}^{l} P^2_j(x, y)t^{n+j}
\end{aligned} \tag{3.2}$$

where $P^k_j(x, y)(k = 1, 2,\ j = 1, \ldots, l)$ are algebraic polynomials in x and y with undetermined real coefficients. We will describe the procedure for constructing these polynomials.

In order for the displacements (3.2) to satisfy Eqs (1.1), the polynomials $P^k_j(x, y)$ must satisfy relations of the form (2.2). Hence, when $k = l, l-1$ the following representations hold

$$\begin{aligned}
P^1_k(x, y) + iP^2_k(x, y) &= \operatorname{Re}\left[\tilde{P}^1_k(z) - \frac{\bar{z}}{2\kappa}[\tilde{P}^1_k(z) + \tilde{P}^2_k(z)]'\right] +\\
&+ i\operatorname{Im}\left[\tilde{P}^2_k(z) + \frac{\bar{z}}{2\kappa}[\tilde{P}^1_k(z) + \tilde{P}^2_k(z)]'\right]
\end{aligned} \tag{3.3}$$

where $\tilde{P}^j_k(z)(j = 1, 2)$ are algebraic polynomials of the complex variable $z = x + iy$ with arbitrary complex coefficients. When $k < l-1$ representation (3.3) is supplemented by the term

$$(k+1)(k+2)S[P^1_{k+2}(x, y) + iP^2_{k+2}(x, y)]$$

where S is an operator, defined by the second formula of (2.3).

It is obvious that the coefficient for each additional power t higher than the n-th in representation (3.2) occurs in the solution of the two additional complex polynomials, i.e., the two sets of arbitrary complex parameters. Hence, the function $\tilde{u} + i\tilde{\upsilon}$ contains two $2l$ sets of arbitrary complex parameters or $4l$ sets of arbitrary real parameters.

By satisfying the condition of zero boundary displacements at the nodal points $t_j (j=0,\ldots,n)$ for $\tilde{u}, \tilde{\upsilon}$, we obtain an expression for the boundary values of the coefficients $\tilde{u}_k,\ \tilde{\upsilon}_k$ in terms of the parameters introduced

$$[\tilde{u}_k(x, y) + i\tilde{\upsilon}_k(x, y)]\big|_{(x, y)\in C} = -\sum_{p=1}^{l} \alpha^k_p[P^1_p(x, y) + iP^2_p(x, y)]\big|_{(x, y)\in C} \tag{3.4}$$

where α^k_p is a fraction, the denominator of which is a Vandermond determinant with powers (from the zeroth to the n-th) of the numbers t_j ($j=0, 1, \ldots, n$), while the numerator is the same determinant in which columns with the numbers t^k_j are replaced by columns with the numbers t^{n+p}_j.

Further, the coefficients $\tilde{u}_k(x, y), \tilde{\upsilon}_k(x, y)(k = 0, 1, \ldots, n)$ are found using the scheme for establishing the coefficients $u_k(x, y), \upsilon_k(x, y)(k = 0, 1, \ldots, n)$ from Section 2, beginning with the n-th number. Note that, in view of boundary conditions (3.4), the polyharmonic functions $\tilde{u}_k(x, y), \tilde{\upsilon}_k(x, y)$ $(k = 0, 1, \ldots, n)$ will depend linearly on the undetermined real coefficients contained in the polynomials $P^1_p(x, y)$ and $P^2_p(x, y)(p = 1, \ldots l)$.

It can be verified that any polynomial $P(x, y)$ is a polyharmonic function, since a natural number n exists, such that

$$\frac{\partial^{2n}P}{\partial z^n\partial\bar{z}^n} = \frac{1}{2^{2n}}\Delta^n P \equiv 0$$

Consequently, according to representations (3.2) the functions $\tilde{u}(x, y, t_0), \tilde{\upsilon}(x, y, t_0), \partial_t\tilde{u}(x, y, t_0), \partial_t\tilde{\upsilon}(x, y, t_0)$ will also be polyharmonic functions and will depend linearly on all the undetermined real coefficients, contained in the polynomials $P^k_p(x, y)(k = 1, 2, p = 1, \ldots, l)$. Suppose the number of such coefficients is $2m+2m'$. Then, by satisfying relations (3.1), we obtain a system of $2(m+m')$ equations that are linear in $2(m+m')$ unknown real coefficients.

We will illustrate this, supplementing the example considered at the end of Section 2, by specifying a single interpolation node for the initial displacement and a single interpolation node for the initial velocity. In order to interpolate the initial velocity and the initial displacement at all points of the unit circle, we will obtain the initial displacement in the form

$$\tilde{w}(x, y, t) = \tilde{w}_0(x, y) + \tilde{w}_1(x, y)t + \tilde{w}_2(x, y)t^2 + (a+ib)t^3 + (c+id)t^4$$

By satisfying the zeroth boundary displacements for $t = 0, 1, 2$ we obtain

$$\tilde{w}_0(x, y) = \frac{\rho}{\lambda + 3\mu}[3(a + ib) + 7(c + id)](1 - |z|^2) +$$
$$+ \frac{\rho^2(\lambda + \mu)(c - id)}{4\mu(\lambda + 3\mu)(\lambda + 2\mu)} z^2(1 - |z|^2) -$$
$$- \frac{\rho^2(c + id)(1 - |z|^2)}{\mu(\lambda + 2\mu)}\left[\frac{3(|z|^2 - 3)}{8} + \frac{(\lambda + \mu)^2}{(\lambda + 3\mu)^2}\right]$$

$$\tilde{w}_1(x, y) = 2(a + ib) + 6(c + id) - \frac{3\rho(a + ib)}{\lambda + 3\mu}(1 - |z|^2)$$

$$\tilde{w}_2(x, y) = -[3(a + ib) + 7(c + id)] - \frac{6\rho(c + id)}{\lambda + 3\mu}(1 - |z|^2)$$

It now remains to determine the constants a, b, c and d introduced by satisfying the initial conditions at specified nodal points. The principal determinant of the corresponding system of the form (3.1) will depend on the Lamé coefficients λ and μ, and also on the position of the nodal points. However, noting that quantities of the form $\rho/(\alpha\gamma + \beta\mu)$ are numerically very small for $\alpha = 1, 0$ and $\beta = 1, 2, 3$, we can conclude that the principal determinant is positive, and so the constants a, b, c and d can be found for any specification of the nodal points.

The problem of the solvability of the system is very complex in the general case since the value of the principal determinant of the system also depends on the form of the region and the specific form of the polynomials $P_p^k(x, y)(k = 1, 2, p = 1, \ldots, l)$.

We will assume that the corresponding system (3.1) is solvable for arbitrary real constants. We obtain interpolation of the initial conditions using polyharmonic functions. Since each harmonic function occurring in the representation of a polyharmonic function can be approximated as accurately as desired by a polynomial of two variables, to estimate the interpolation error we can use the well-known formula for the residual term when interpolating a function of two variables by polynomials.[2]

The class of polyharmonic functions is a natural generalization of the class of polynomials. Thus, by Stone's theorem[3] any function, continuous in a bounded and closed region D, can be approximated as accurately as desired in a metric of space $C(D)$ of polyharmonic functions. According to existing results[4] the class of polyharmonic functions in a bounded and closed region is closed with respect to uniform convergence. Moreover, it was proved in Ref. 4 that a uniformly converging sequence of polyharmonic functions can be differentiated inside the region an infinite number of times, where the sequences of derivatives will converge uniformly to the corresponding derivative of the limit function.

4. An estimate of the interpolation error

An estimate of the error when the true solution of the second basic plane problem of the dynamics of elastic bodies is replaced by an interpolation solution, due to the linearity of the dynamic equations and the initial and boundary conditions, reduces to estimating the norm of the solution of the problem

$$(\lambda + \mu)\partial_x\theta + \mu\Delta u = \rho\partial_{tt}u, \quad (\lambda + \mu)\partial_y\theta + \mu\Delta \upsilon = \rho\partial_{tt}\upsilon$$
$$\theta = \partial_x u + \partial_y \upsilon$$

with boundary conditions

$$(u(x, y, t), \upsilon(x, y, t))|_{(x, y) \in C} = (r_1(s, t), r_2(s, t)), \quad t \in [t_0, T], \quad s \in [0, l]$$

$(r_k(s, t_j) = 0)$, $k = 1,2$, $j = 0,1,\ldots,n$) and initial conditions

$$(u(x, y, t_0), \upsilon(x, y, t_0)) = (h_1(x, y), h_2(x, y))$$
$$(\partial_t u(x, y, t_0), \partial_t \upsilon(x, y, t_0)) = (V_1(x, y), V_2(x, y)), \quad (x, y) \in D$$
$$(h_k(x, y)|_{(x, y) \in C} = V_k(x, y)|_{(x, y) \in C} = 0, \ k = 1, 2)$$

We will assume that the initial conditions of the original problem are specified by polyharmonic functions which is natural due to the possibility, mentioned in the previous section, of approximating continuous functions by polyharmonic functions. As a result of this and the property of the convergence of the derivatives of polyharmonic functions, mentioned in the previous section, we can assume that $h_k(x, y)$, $V_k(x, y)$, $\partial_x h_k$, $\partial_y h_k$ $(k = 1,2)$ are polyharmonic functions bounded in $C(D)$. We will also assume that

$$\left\|\sqrt{r_1^2 + r_2^2}\right\|_{C([0, l] \times [t_0, T])} < \epsilon_0, \quad \left\|\sqrt{V_1^2 + V_2^2}\right\|_{L_2(D)} < \epsilon_1$$
$$\|\partial_x h_1\|_{L_2(D)} < \epsilon_2, \quad \|\partial_y h_2\|_{L_2(D)} < \epsilon_3, \quad \|\partial_y h_1 + \partial_x h_2\|_{L_2(D)} < \epsilon_4$$

We will also suppose that the true solution has second derivatives of the components of the displacements with respect to x and with respect y, continuous up to C, for all $t\in[t_0, T]$, and that the derivatives with respect to t of the above-mentioned second derivatives with respect to x and with respect to y are also integrable with a modulus along C. Hence, we can assume that the components of the stress tensor, obtained on the basis of the displacements (u, v) are functions that are continuous on C, the vector of the normal stresses σ_n is a vector function, continuous along C, and its derivative with respect to t is a vector function $\partial_t\sigma_n$ is integrable with a modulus along C.

Using the well-known approach in Ref. 1, we will consider the following relation, obtained using Green's formula

$$\oint_C(\boldsymbol{\sigma}_n, \partial_t\mathbf{a})ds = \frac{1}{2}\partial_t\left[\iint_D[\rho((\partial_t u)^2+(\partial_t \upsilon)^2)+2\mu((\partial_x u)^2+\right.$$
$$\left.+(\partial_y \upsilon)^2)+\lambda(\partial_x u+\partial_y \upsilon)^2+\mu(\partial_y u+\partial_x \upsilon)^2]dxdy\right]$$
$$(\partial_t\mathbf{a}=(\partial_t r_1, \partial_t r_2)). \tag{4.1}$$

We integrate equality (4.1) over t from t_0 to t and we write the integral on the right-hand side of (4.1) in the form

$$\iint_D[\rho(V_1^2+V_2^2)+2\mu((\partial_x h_1)^2+(\partial_y h_2)^2)+\lambda(\partial_x h_1+\partial_y h_2)^2+$$
$$+\mu(\partial_y h_1+\partial_x h_2)^2]dxdy+2\oint_C(\boldsymbol{\sigma}_n, \mathbf{a})ds-2\int_{t_0}^{t}dt\oint_C(\partial_t\boldsymbol{\sigma}_n, \mathbf{a})ds$$

Since the integral on the right-hand side of (4.1) is a linear combination of squares of norms in the space $L_2(D)$ of several functions, the following estimate holds

$$\rho\left\|\sqrt{(\partial_t u)^2+(\partial_t \upsilon)^2}\right\|^2+2\mu(\|\partial_x u\|^2+\|\partial_y \upsilon\|^2)+\lambda\|\partial_x u+\partial_y \upsilon\|^2+$$
$$+\mu\|\partial_y u+\partial_x \upsilon\|^2\le\rho\epsilon_1^2+2\mu(\epsilon_2^2+\epsilon_3^2)+\lambda(\epsilon_2+\epsilon_3)^2+\mu\epsilon_4^2+$$
$$+2\epsilon_0(\max_{t\in[t_0,T]}\oint|\boldsymbol{\sigma}_n|ds+(T-t_0)\max_{t\in[t_0,T]}\oint|\partial_t\boldsymbol{\sigma}_n|ds) = \delta^2$$

On the left-hand side of the last inequality and henceforth $\|\cdot\|$ means $\|\cdot\|_{L_2(D)}$.

Hence, we can estimate the norms of the differences in the components of the stress tensors corresponding to the true and interpolation solutions of the second basic plane problem of the dynamics of elastic bodies, denoting them by σ_{ij} $(i,j=1,2)$. We obtain

$$\|\sigma_{11}\|\le(\sqrt{\lambda}+\sqrt{2\mu})\delta,\quad \|\sigma_{22}\|\le(\sqrt{\lambda}+\sqrt{2\mu})\delta,\quad \|\sigma_{12}\|\le\sqrt{\mu}\delta$$

5. A spline-interpolation solution

This method of constructing an interpolation solution can be used to construct a spline-interpolation solution. Here the section $[t_0,T]$ is divided into sections $[t_j, t_{j+1}]$. In each such section we seek a solution which, when $(x, y, t)\in D\times(t_{j-1}, t_j)$, satisfies Eqs (1.1), satisfies the specified boundary displacements when $t=t_{j+1}$, and which satisfies the given boundary displacements and velocities on the boundary and at a finite number of inner points of the region D when $t=t_j$. For $j=0$ we use the specified initial conditions on the boundary and at a finite number of inner points. For $j>0$, instead of the initial conditions we use the displacements and velocities on the boundary and at a finite number of inner points of the region, obtained at the previous stage when $t=t_{j-1}$.

The solution for each section $[t_j, t_{j+1}]$ is most conveniently sought in the form $\breve{u}+i\breve{v}+\tilde{u}+i\tilde{v}$, where

$$\breve{u}+i\breve{\upsilon} = \sum_{k=0}^{2}(u_k(x, y)+i\upsilon_k(x, y))t^k$$

satisfies Eqs (1.1), with the specified boundary displacements when $t=t_j$, t_{j+1}, and the assigned boundary velocities when $t=t_j$. Here, to determine the boundary values of the coefficients $\breve{u}_k(x, y)$ and $\breve{v}_k(x, y)$ $(k=0,\ 1,\ 2)$ we obtain a system, the principal determinant of which is equal to $-(t_{j+1}-t_j)^2$ (unlike the Vandermond determinant in Section 2).

In turn,

$$\tilde{u}+i\tilde{\upsilon} = \sum_{k=0}^{2}(\tilde{u}_k(x, y)+i\tilde{\upsilon}_k(x, y))t^k+\sum_{p=1}^{l}(P_p^1(x, y)+iP_p^2(x, y))t^{2+p}$$

satisfies Eqs (1.1), the zero boundary displacements when $t = t_j$, t_{j+1}, and the zero velocities when $t = t_j$, and, because of the presence of the polynomials $P_p^k(x, y)$, it interpolates the displacements and velocities in the region D in accordance with the values specified or obtained at a finite number of inner nodal points of the region when $t = t_j$.

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